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Introduction

- In this topic, we will
  - Describe a small change to the Jacobi method
  - Observe that this small change:
    - Results in faster convergence
    - Guarantees convergence for another class of matrices
    - Requires no additional work

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The Jacobi method

- The Jacobi method approximate the solution to

$$A\mathbf{u} = \mathbf{v}$$

by iterating

$$\mathbf{u}_{k+1} \leftarrow A_{\text{diag}}^{-1}\left(\mathbf{v} - A_{\text{off}}\mathbf{u}_k\right)$$

  – Normally, you think of this as calculating the entire vector

The Gauss-Seidel method

- Gauss and Seidel realized that we can calculate the entries of $\mathbf{u}_{k+1}$ one at a time, and to then use this updated entry when calculating the next entry
  – The next approximation begins with

$$\mathbf{u}_{k+1} \leftarrow \mathbf{u}_k$$

  – We then update the i^{th} entry of $\mathbf{u}_{k+1}$ one at a time:

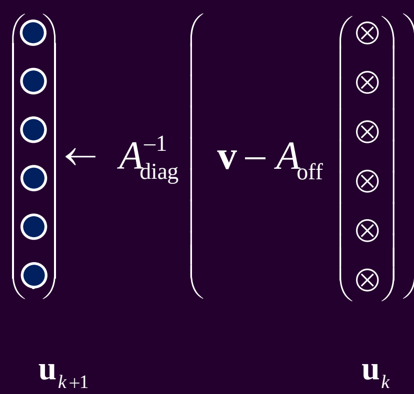
$$\mathbf{u}_{k+1;\,i} \leftarrow \frac{1}{a_{i,i}}\left(v_i - A_{\text{off; i,}\cdots}\mathbf{u}_{k+1}\right)$$

  where $A_{\text{off; i,}\cdots}$ is the i^{th} row of the matrix A_{off}

  – Note that when we calculate $\mathbf{u}_{k+1;\,3}$, the first two entries have already been updated

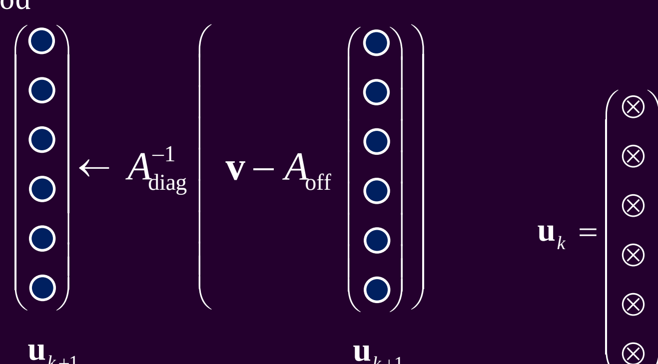
The Gauss-Seidel method

- Visually, this is what occurs in solving $A\mathbf{u} = \mathbf{v}$ with the Jacobi method



The Gauss-Seidel method

- Visually, this is what occurs in solving $A\mathbf{u} = \mathbf{v}$ with the Gauss-Seidel method



The Gauss-Seidel method

- The Gauss-Seidel method continues to converge if the matrix is strictly diagonally dominant
  - It actually speeds up convergence
- Unlike the Jacobi method, the Gauss-Seidel method is also guaranteed to converge if the matrix is symmetric and positive definite
  - Such a matrix has all positive eigenvalues

Implementation

- Suppose we are solving $A\mathbf{u} = \mathbf{v}$ and $\mathbf{u}$ is the current approximation:

```
for k = 1:max_iterations
    u_old = u;

    for i = 1:n
        u(i) = v(i);

        for j = [1:i-1, i+1:n]
            u(i) = u(i) - A(i,j)*u(j);
        end

        u(i) = u(i)/A(i,i);
    end

    if norm( u - u_old ) < eps_step
        return; // returns 'u'
    end
end
```

This was:

```
u(i) = u(i) - a(i,j)*u_old(j);
```

Summary

- Following this topic, you now
  - Have an understanding of the Gauss-Seidel method
  - Understand that we can use more recent vector entries when calculating subsequent vector entries
  - Understand that this allows for must faster convergence
  - Are aware this works for a greater range of matrices at no additional cost

The Gauss-Seidel method

References

[1] https://en.wikipedia.org/wiki/Gauss-Seidel_method

Acknowledgments

None so far.

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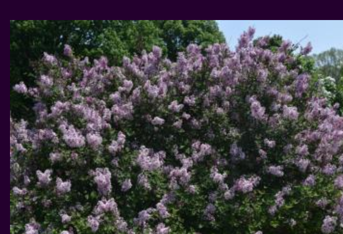
Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
Mathematical equations are prepared in MathType by Design Science, Inc.
Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.



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